

Traces of Subharmonic Functions to Fractal Sets

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June 23, 1999

Abstract

We study traces of a class of subharmonic functions to Ahlfors regular subsets of \mathbb{C}^n . In particular, we establish for the traces a generalized BMO-property and the reverse Hölder inequality.

1. Introduction.

1.1. A compact subset $K \subset \mathbb{R}^n$ is said to be (Ahlfors) *d-regular* if there is a positive number a such that for any $x \in K$, $0 < t \leq \text{diam}(K)$

$$\mathcal{H}^d(K \cap \mathbb{D}(x, t)) \leq at^d. \quad (1.1)$$

Here $\mathcal{H}^d(\omega)$ denotes the d -Hausdorff measure of ω . This class will be denoted by $\mathcal{A}(d, a)$.

A compact subset $K \in \mathcal{A}(d, a)$ is said to be a *d-set* if there is a positive number b such that for any $x \in K$, $0 < t \leq \text{diam}(K)$

$$bt^d \leq \mathcal{H}^d(K \cap \mathbb{D}(x, t)). \quad (1.2)$$

Denote this class by $\mathcal{A}(d, a, b)$.

The purpose of this paper is to study traces of subharmonic functions to d -sets. Let us recall that the class of d -sets, in particular, contains Lipschitz d -manifolds (with d integer), Cantor type sets and self-similar sets (with arbitrary d), see, e.g., [JW], p. 29 and [M], sect. 4.13.

Let us, first, formulate our results in \mathbb{C} . In the sequel we denote $\mathbb{D}_s := \{z \in \mathbb{C} : |z| < s\}$ and $\mathbb{D}(x, t) := \{z \in \mathbb{C} : |z - x| < t\}$.

Assume that f is a subharmonic in \mathbb{D}_1 function satisfying

$$\sup_{\mathbb{D}_1} f \leq M_1 \quad \text{and} \quad \sup_{\mathbb{D}_r} f \geq M_2 \quad (r < 1). \quad (1.3)$$

*1991 *Mathematics Subject Classification*. Primary 31B05. Secondary 46E30.
Key words and phrases. Plurisubharmonic function, d -regular set, BMO-function.

Theorem 1.1 *Let $\omega \subset \mathbb{D}(x, t)$ be a compact set of $\mathcal{A}(d, a)$ satisfying $\mathcal{H}^d(\omega) \geq \epsilon > 0$. Assume that $\mathbb{D}(x, t/r) \subset \mathbb{D}_r$. Then there is a constant $c = c(r) > 0$ such that inequality*

$$\sup_{\mathbb{D}(x, t)} f \leq \sup_{\omega} f + (M_1 - M_2)c \log \frac{4eta^{1/d}}{r(d\epsilon)^{1/d}}$$

holds for any subharmonic f satisfying (1.3).

We use Theorem 1.1 to establish a generalized BMO-property and the reverse Hölder inequality for traces of subharmonic functions to d -sets. Let us recall

Definition 1.2 *Let X be a complete metric space equipped with a regular Borel measure μ . A locally integrable on X function f belongs to $BMO(X, \mu)$ if*

$$|f|_* := \sup \left\{ \frac{1}{\mu(B)} \int_B |f - f_B| d\mu \right\} < \infty;$$

here supremum is taken over all metric balls $B \subset X$ and $f_B = \frac{1}{\mu(B)} \int_B f d\mu$.

In order to formulate the next two results consider a compact d -set $K \subset \mathbb{C}$. Assume that f is a subharmonic function defined in an open neighbourhood of K .

Theorem 1.3 *Restriction $f|_K$ belongs to $BMO(K, \mathcal{H}^d)$.*

Theorem 1.4 *For any $K_{x,t} := \mathbb{D}(x, t) \cap K$, $x \in K$, $t > 0$, and any $1 \leq p \leq \infty$ the inequality*

$$\left(\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^{pf} d\mathcal{H}^d \right)^{1/p} \leq C(K, f, d) \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^f d\mathcal{H}^d \quad (1.4)$$

holds.

1.2. In this section we consider multidimensional generalizations of the previous results for plurisubharmonic (*psh*) functions in \mathbb{C}^n , $n \geq 2$. Note that it is impossible to obtain in this situation the results similar to those in \mathbb{C} . Indeed, let $N := \{z \in \mathbb{C}^n : f(z) = 0\} \neq \emptyset$ for a nontrivial holomorphic function f . Let $K = K_1 \cup K_2$ be a connected set where K_1, K_2 are compact $(2n - 2)$ -Lipschitz manifolds such that $K_1 \subset N$ and $K_2 \not\subset N$. Then the psh function $\log |f|$ equals $-\infty$ on K_1 and $\sup_K \log |f| > -\infty$. Hence the results similar to Theorems 1.1, 1.3 and 1.4 are not true in this case.

So we restrict ourselves to a special class of psh functions whose local behavior is similar to that of subharmonic functions in \mathbb{C} .

Consider a family $F = \{f_1, \dots, f_n\}$ of holomorphic functions defined in Euclidean ball $B_r \subset \mathbb{C}^n$ of radius $r > 1/2$ centered at 0. We will prove the results for psh function $u = \log |F|$ where $|F|^2 := |f_1|^2 + \dots + |f_n|^2$. Assume that

$$\sup_{B_r} u \leq 0 \quad \text{and} \quad \inf_{\partial B_{1/2}} u \geq -M \quad (1.5)$$

for some $M > 0$. The latter condition means that the system $F = 0$ has only discrete zeros in a neighbourhood of the closed ball $\overline{B_{1/2}}$.

Definition 1.5 Let h be a nonnegative analytic function with a finite number of zeros defined on an open set $U \subset \mathbb{R}^N$. A zero x of h is said to be elliptic if the Taylor expansion of f at x has the following form

$$h(x + t\omega) = t^d f(\omega) + o(t^d)$$

with

$$\inf_{S^{N-1}} f > 0 .$$

Here $t > 0$ and ω belongs to the unit sphere S^{N-1} of \mathbb{R}^N .

Assume that F satisfies condition (1.5) and all zeros of $|F|^2$ are elliptic. Let k be the number of zeros of $|F|^2$ in $B_{1/2}$ counting with their multiplicities. (By multiplicity we mean the degree of map F at zero.)

Theorem 1.6 Let $\omega \subset B(x, t)$ be a compact set of $\mathcal{A}(d, a)$ and $\mathcal{H}^d(\omega) \geq \epsilon > 0$. Assume that $B(x, 4r^2t) \subset B_{1/2}$. Then there is a constant $c = c(r, F) > 0$ such that

$$\sup_{B(x, t)} \log |F| \leq \sup_{\omega} \log |F| + k \log \frac{16erta^{1/d}}{c(d\epsilon)^{1/d}} .$$

The following theorem gives the results similar to those of Theorems 1.3 and 1.4.

Theorem 1.7 Assume that a compact set $K \subset \mathbb{C}^n$ belongs to $\mathcal{A}(d, a, b)$. Then under assumptions of Theorem 1.6, $|F|$ satisfies the reverse Hölder inequality in each ball $B(x, t) \cap K$ and $\log |F| \in BMO(K, \mathcal{H}^s)$.

2. Abstract Version of Cartan's Lemma.

Our proofs are based on estimates for psh functions which generalize well-known Cartan's Lemma for polynomials (see [Ca]). We need a version of the generalized Cartan's Lemma proved by Gorin (see [GK]).

Let X be a complete metric space and let μ be a finite Borel measure on X . We consider a continuous, strictly increasing, nonnegative function ϕ on $[0, +\infty[$, $\phi(0) = 0$, $\lim_{x \rightarrow \infty} \phi(x) > \mu(X)$. The function ϕ will be called a *majorant*.

For each point $x \in X$ we set $\tau(x) = \sup\{t : \mu(B(x, t)) \geq \phi(t)\}$, where $B(x, t)$ is the closed ball in X with center x and radius t . It is easy to see that $\mu(B(x, \tau(x))) = \phi(\tau(x))$ and $\sup_x \tau(x) \leq \phi^{-1}(\mu(X)) < \infty$.

A point $x \in X$ is said to be *regular* (with respect to μ and ϕ) if $\tau(x) = 0$, i.e., $\mu(B(x, t)) < \phi(t)$ for all $t > 0$. The next result shows that the set of regular points is sufficiently large for an arbitrary majorant ϕ .

Lemma 2.1 (Gorin) Let $0 < \gamma < 1/2$. There exists a sequence of balls $B_k = B(x_k, t_k)$, $k = 1, 2, \dots$, which collectively cover all the irregular points and which are such that $\sum_{k \geq 1} \phi(\gamma t_k) \leq \mu(X)$ (i.e., $t_k \rightarrow 0$).

For the sake of completeness we present Gorin's proof of the lemma.

Proof. Let $0 < \alpha < 1$, $\beta > 2$ but $\gamma < \alpha/\beta$. We set $B_0 = \emptyset$ and assume that the balls B_0, \dots, B_{k-1} have been constructed. If $\tau_k = \sup\{\tau(x) : x \notin B_0 \cup \dots \cup B_{k-1}\}$, then there exists a point $x_k \notin B_0 \cup \dots \cup B_{k-1}$, such that $\tau(x_k) \geq \alpha\tau_k$. We set $t_k = \beta\tau_k$ and $B_k = B(x_k, t_k)$. Clearly, the sequence τ_k (and thus also t_k) does not increase. The balls $B(x_k, \tau_k)$ are pairwise disjoint. Indeed, if $l > k$, then $x_l \notin B_k$, i.e., the distance between x_l and x_k is greater than $\beta\tau_k > 2\tau_k \geq \tau_k + \tau_l$. Then,

$$\sum_{k=1}^{\infty} \phi(\gamma t_k) \leq \sum_{k=1}^{\infty} \phi(\alpha\tau_k) \leq \sum_{k=1}^{\infty} \phi(\tau(x_k)) = \sum_{k=1}^{\infty} \mu(B(x_k, \tau_k)) \leq \mu(X) ;$$

consequently, $\tau_k \rightarrow 0$, i.e., for each point x , not belonging to the union of the balls B_k , $\tau(x) = 0$, x is a regular point. In addition, $t_k = \beta\tau_k \rightarrow 0$. \square

Remark 2.2 If X is a locally compact metric space then one can take $\gamma = 1/2$ (for similar arguments see, e.g., [L], Th. 11.2.3).

We now apply Lemma 2.1 to obtain estimates for logarithmic potentials of measures.

Assume that X is a locally compact metric space with metric $d(., .)$.

Theorem 2.3 *Let*

$$u(z) = \int_X \log d(x, \xi) d\mu(\xi)$$

where μ is a Borel measure, $\mu(X) = k < \infty$.

Given $H > 0, d > 0$ there exists a system of metric balls such that

$$\sum r_j^d \leq \frac{(2H)^d}{d} \tag{2.1}$$

where r_j are radii of these balls, and

$$u(z) \geq k \log \frac{H}{e}$$

everywhere outside these balls.

Proof. Let $\phi(t) = (pt)^d$ be a majorant with $p = \frac{(kd)^{1/d}}{H}$. We cover all irregular points of X by balls according to Gorin's Lemma 2.1 and Remark 2.2. It remains to estimate the potential u outside of these balls, i.e., at any regular point z . Let $n(t; z) = \mu(\{\xi : d(z, \xi) \leq t\})$. Clearly, for any $N \geq \max\{1, H\}$

$$u(z) \geq \int_{d(z, \xi) \leq N} \log d(z, \xi) d\mu(\xi) = \int_0^N \log t \, dn(t; z) = n(t; z) \log t|_0^N - \int_0^N \frac{n(t; z)}{t} dt.$$

Since $n(t; z) < (pt)^d$, we then have

$$u(z) \geq n(N; z) \log N - \int_0^N \frac{n(t; z)}{t} dt.$$

In addition, $n(t; z) \leq n(N; z)$ for $t \leq N$. Therefore,

$$\begin{aligned} u(z) &\geq n(N; z) \log N - \int_0^H \frac{(pt)^d}{t} dt - \int_H^N \frac{n(N; z)}{t} dt = \\ &n(N; z) \log N - \frac{(pH)^d}{d} - n(N; z) \log N + n(N; z) \log H = -k + n(N; z) \log H \end{aligned}$$

Letting here $N \rightarrow \infty$ and taking into account that $\lim_{N \rightarrow \infty} n(N; z) = k$ we obtain the required result. \square

3. Proofs of Theorems 1.1, 1.3 and 1.4.

Proof of Theorem 1.1. We begin with

Proposition 3.1 *Let u be a nonpositive subharmonic function on \mathbb{D}_1 satisfying*

$$\sup_{\mathbb{D}_r} u \geq -1 \quad \text{for some } r < 1.$$

Then for any $H > 0, d > 0$ there is a set of disks such that

$$\sum r_j^d \leq \frac{(2H)^d}{d}, \quad (3.1)$$

where r_j are radii of these disks, and

$$u(z) \geq c \log \frac{H}{e}$$

outside these disks in \mathbb{D}_r . Here $c = c(r) > 0$ depends on r only.

Proof. Let κ be a nonnegative radial C^∞ -function on \mathbb{C} satisfying

$$\int \int_{\mathbb{C}} \kappa(x, y) dx dy = 1 \quad \text{and} \quad \text{supp}(\kappa) \subset \mathbb{D}_1 \quad (z = x + iy). \quad (3.2)$$

Let u_k denote the function defined on $\mathbb{D}_{1-1/k}$ by

$$u_k(w) := \int \int_{\mathbb{C}} \kappa(x, y) u(w - z/k) dx dy. \quad (3.3)$$

It is well known, see, e.g., [K], Theorem 2.9.2, that u_k is subharmonic on $\mathbb{D}_{1-1/k}$ of the class C^∞ and that $u_k(w)$ monotonically decreases and tends to $u(w)$ for each $w \in \mathbb{D}_1$ as $k \rightarrow \infty$. Let $K := \{z \in \mathbb{D}_1 : \frac{1+r}{2} \leq |z| \leq \frac{3+r}{4}\}$ be an annulus in \mathbb{D}_1 and $k \geq k_0 = [\frac{8}{1-r}] + 1$. We are based on the following result (see, e.g., [Br], Lemma 2.3).

There are a constant $A = A(r) > 0$ and numbers $t_k, k \geq k_0$, satisfying $\frac{1+r}{2} \leq t_k \leq \frac{3+r}{4}$ such that $u_k(z) \geq -A$ for any $z \in \mathbb{C}, |z| = t_k$.

Then we can construct functions f_k subharmonic on \mathbb{C} by

$$f_k(z) := \begin{cases} u_k(z) & (z \in \mathbb{D}_{t_k}); \\ \max \left\{ u_k(z), \frac{-2A \log |z|}{\log t_k} \right\} & (z \in \mathbb{D}_1 \setminus \mathbb{D}_{t_k}); \\ \frac{-2A \log |z|}{\log t_k} & (z \in \mathbb{C} \setminus \mathbb{D}_1). \end{cases}$$

Without loss of generality we may assume that $t_k \rightarrow t \in [\frac{1+r}{2}, \frac{3+r}{4}]$ as $k \rightarrow \infty$. Finally, define

$$f(z) = (\overline{\lim_{k \rightarrow \infty}} f_k(z))^*,$$

where g^* denotes upper semicontinuous regularization of g . Then f is subharmonic in \mathbb{C} satisfying

$$f(z) = u(z) \quad (z \in \mathbb{D}_1) \quad \text{and} \quad f(z) = \frac{-2A \log |z|}{\log t} \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}_1}).$$

Consider now $\mu = \Delta f$. Then μ is a finite Borel measure on \mathbb{C} supported in $\overline{\mathbb{D}_1}$. According to F. Riesz's theorem (see, e.g., [HK], Th. 3.9)

$$\tilde{f}(z) := \frac{1}{2\pi} \int \int_{\mathbb{C}} \log |z - \xi| d\mu(\xi)$$

is subharmonic in \mathbb{C} and satisfies $\Delta \tilde{f} = \Delta f = \mu$. Thus $h = \tilde{f} - f$ is a real-valued harmonic in \mathbb{C} function. Moreover, h goes to infinity as $(\frac{\mu(\mathbb{C})}{2\pi} - \frac{-2A}{\log t}) \log |z|$. This immediately implies $h = 0$ and $\frac{\mu(\mathbb{C})}{2\pi} = \frac{-2A}{\log t}$. Now according to Theorem 2.3 applied to $f(= \tilde{f})$, for any $H > 0, d > 0$ there is a system of disks with radii r_j satisfying $\sum r_j^d \leq \frac{(2H)^d}{d}$ such that

$$f \geq \frac{-2A}{\log t} \log \frac{H}{e} \geq \frac{-2A}{\log r} \log \frac{H}{e}$$

outside these disks. It remains to set $c = \frac{-2A}{\log r}$.

The proof of the proposition is complete. \square

Assume now that f is subharmonic and satisfies (1.3). Then by the main theorem in [Br] there is a constant $C = C(r) > 0$ such that the inequality

$$\sup_{\mathbb{D}(x, t/r)} f \leq C(M_1 - M_2) + \sup_{\mathbb{D}(x, t)} f$$

holds for any pair of disks $\mathbb{D}(x, t) \subset \mathbb{D}(x, t/r) \subset \mathbb{D}_r$. Applying inequality of Proposition 3.1 to the function

$$u(z) = \frac{f(tz/r) - \sup_{\mathbb{D}(x, t/r)} f}{C(M_1 - M_2)} \quad (z \in \mathbb{D}_1)$$

and then going back to f we obtain the following

Proposition 3.2 *There is a constant $c = c(r) > 0$ such that for any disk $\mathbb{D}(x, t)$ satisfying $\mathbb{D}(x, t) \subset \mathbb{D}(x, t/r) \subset \mathbb{D}_r$ and any $H > 0, d > 0$ there is a system of disks such that*

$$\sum r_j^d \leq \frac{(2tH/r)^d}{d},$$

where r_j are radii of these disks, and

$$f(z) \geq \sup_{\mathbb{D}(x, t)} f + c(M_1 - M_2) \log \frac{H}{e}$$

outside these disks in $\mathbb{D}(x, t)$.

Remark 3.3 A particular case of Proposition 3.2 for functions $u = \log |f|$ with holomorphic f and for $d = 1$ was proved in [L].

We proceed to the proof of Theorem 1.1. First we show that ω can not be covered by a system of disks such that

$$\sum r_j^d \leq \frac{(1 - 1/n)\epsilon}{2^d a} \quad (n \geq 1) \quad (3.4)$$

where r_j are radii of these disks. Assume to the contrary that there exists a system of disks $\{\mathbb{D}(x_j, r_j)\}$ whose radii satisfy (3.4) which covers ω . For any x_j choose $y_j \in \omega$ so that $|x_j - y_j| \leq r_j$. Then the system of disks $\{\mathbb{D}(y_j, 2r_j)\}$ also covers ω . Since $\omega \in \mathcal{A}(d, a)$, we obtain inequality

$$\mathcal{H}^d(\omega) \leq \sum \mathcal{H}^d(\omega \cap \mathbb{D}(y_j, 2r_j)) \leq 2^d a \sum r_j^d < \epsilon$$

which contradicts to $\mathcal{H}^d(\omega) \geq \epsilon$.

We now apply Proposition 3.2 with $H_n = \frac{(d(1-1/n)\epsilon)^{1/d} r}{4ta^{1/d}}$. Since any system of disks with $\sum r_j^d \leq \frac{(2tH_n/r)^d}{d}$ can not cover ω , Proposition 3.2 implies that there is a point $x_n \in \omega$ such that

$$\sup_{\omega} f \geq f(x_n) \geq \sup_{\mathbb{D}(x, t)} f + c(M_1 - M_2) \log \frac{H_n}{e}$$

Letting $n \rightarrow \infty$ we get the required inequality.

Theorem 1.1 is proved. \square

Our next result shows that d -regularity is a necessary condition for the set to satisfy the inequality of Theorem 1.1.

Proposition 3.4 *Let $K \subset \mathbb{D}_{1/2}$ be a compact set with $\mathcal{H}^d(K) < \infty$. Assume that the inequality*

$$\sup_{\mathbb{D}(x, t)} f \leq \sup_{\omega} f + L + C \log \frac{t}{\epsilon^{1/d}}$$

holds for any $\omega \subset K \cap \mathbb{D}(x, t) \subset \mathbb{D}(x, 3t/2) \subset \mathbb{D}_{2/3}$, $x \in K$, with $\mathcal{H}^d(\omega) = \epsilon$ and any f subharmonic in \mathbb{D}_1 satisfying (1.3) with $r = 2/3$ and some M_1, M_2 . Here L and $C > 0$ depend on K, d, M_1, M_2 . Then $K \in \mathcal{A}(d, c)$ for some $c > 0$.

Proof. For any $f, \omega, t \leq 1/9$ satisfying assumptions of the proposition the inequality

$$-C \log \frac{t}{\epsilon^{1/d}} \leq \sup_{\mathbb{D}(x,t)} f - \sup_{\omega} f - C \log \frac{t}{\epsilon^{1/d}} \leq L < \infty$$

holds. For a point $x \in K$ we set $f_x(z) = \log |z - x|$ and $\epsilon_t := \mathcal{H}^d(\mathbb{D}(x, t) \cap K)$. Clearly the family $\{f_x\}$ satisfies inequality (1.3) with $r = 2/3$, $M_1 = 3/2$ and $M_2 = 1/6$. Then the above inequality applied to f_x gets

$$L \geq -C \log \frac{t}{\epsilon_t^{1/d}},$$

that is equivalent to $\epsilon_t \leq \tilde{L} t^d$ for $\tilde{L} = e^{\frac{dL}{C}}$. So we checked the definition of d -regularity for $t \leq 1/9$. For $t > 1/9$ the inequality is obvious. \square

Assume that f satisfies (1.3) and $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d, a)$. For a pair $\mathbb{D}(x, t) \subset \mathbb{D}(x, t/r) (\subset \mathbb{D}_r)$ we set $K_{x,t} := \mathbb{D}(x, t) \cap K$ and $f_{x,t} = \sup_{\mathbb{D}(x,t)} f$. Further, set $f' = f_{x,t} - f$. In the proofs of Theorem 1.3 and 1.4 we use

Lemma 3.5 *Let $D_{f'}(\lambda) = \mathcal{H}^d\{y \in K_{x,t} : f'(y) \geq \lambda\}$ be the distribution function of f' . Then*

$$D_{f'}(\lambda) \leq \frac{(4et)^d a}{r^d d} e^{-\lambda d / (c(M_1 - M_2))}. \quad (3.5)$$

Proof. The proof follows straightforwardly from the inequality of Theorem 1.1 where we choose $\omega := D_{f'}(\lambda)$. We leave the details to the reader. \square

Proof of Theorem 1.3. First, we prove a local version of the theorem. Assume that $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d, a, b)$ and $f, \mathbb{D}(x, t)$ satisfy conditions of Lemma 3.5. From inequality 3.5 it follows

$$\begin{aligned} \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f' d\mathcal{H}^d &\leq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^\infty D_{f'}(x) dx \leq \frac{1}{bt^d} \frac{c(M_1 - M_2)}{d} \frac{(4et)^d a}{r^d d} = \\ &= \frac{ca(4e)^d (M_1 - M_2)}{br^d d^2}. \end{aligned} \quad (3.6)$$

Now we have

$$\begin{aligned} \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |f - f_{K_{x,t}}| d\mathcal{H}^d &\leq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |(f - f_{x,t}) - (f - f_{x,t})_{K_{x,t}}| d\mathcal{H}^d \leq \\ &\leq \frac{2}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f' d\mathcal{H}^d \leq \frac{2ca(4e)^d (M_1 - M_2)}{br^d d^2}. \end{aligned}$$

This gives the estimate of the BMO-norm in each ball $K(x, t) = \mathbb{D}(x, t) \cap K$ with $\mathbb{D}(x, t) \subset \mathbb{D}(x, t/r) (\subset \mathbb{D}_r)$. In the general case, we cover K by a finite number of open disks $\mathbb{D}(x_i, R)$, $i = 1, \dots, N$ such that f is defined in the union of these disks, the set $\cup_{i=1}^N \mathbb{D}(x_i, R/2)$ also covers K and any disk of radius $\leq R/4$ centered at a point of K belongs to one of $\mathbb{D}(x_i, R/2)$. Then the estimate of the BMO-norm in

any $\mathbb{D}(x, t) \cap K$, $x \in K$, $t \leq R/4$, follows from Theorem 1.1 and inequality (3.6). To estimate BMO-norms for $\mathbb{D}(x, t) \cap K$ with $t \geq R/4$ we write

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |f - f_{K_{x,t}}| d\mathcal{H}^d \leq \frac{4^d}{bR^d} \int_{K_{x,t}} 2|f| d\mathcal{H}^d < C \int_K |f| d\mathcal{H}^d.$$

To complete the proof note that (3.5) implies $\int_K |f| d\mathcal{H}^d < \infty$. \square

We now formulate another corollary of Theorem 1.1.

Corollary 3.6 *Assume that a subharmonic function f defined on \mathbb{C} satisfies*

$$f(z) \leq c' + \log(1 + |z|) \quad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in \mathcal{A}(d, a, b)$. Then $f|_S \in BMO(S, \mathcal{H}^d)$ and the BMO norm $|f|_S|_ \leq \frac{\tilde{c}a}{bd^2}$ with an absolute constant \tilde{c} .*

Proof. For functions f satisfying conditions of the corollary the Bernstein-Walsh inequality

$$\sup_{\mathbb{D}(x, qt)} f \leq \log q + \sup_{\mathbb{D}(x, t)} f \quad (3.7)$$

holds for any $x \in \mathbb{C}$, $t \geq 0$, $q \geq 1$. (The proof is based on the classical Bernstein inequality for polynomials and the polynomial representation of the \mathcal{L} -extremal function of the disk (see, e.g. [K]).) Then the estimate of the BMO-norm in $f|_{\mathbb{D}(x, t) \cap S}$ follows from inequality (3.6) with $r = 1/2$ and $M_1 - M_2 = \log 2$. \square

Proof of Theorem 1.4. As in the proof of Theorem 1.3 we, first, consider a local version of the theorem. Assume that $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d, a, b)$ and $f, \mathbb{D}(x, t)$ satisfy conditions of Lemma 3.5. Denote $g_t = e^{-f'} = e^f / e^{f_{x,t}}$. Consider the distribution function $d_g(\lambda) := \mathcal{H}^d\{y \in K_{x,t} : g_t(y) \leq \lambda\}$. Then from the inequality of Lemma 3.5 for $D_{f'}$ we deduce

$$d_g(\lambda) \leq \frac{(4et)^d a}{r^d d} (\lambda)^{d/(c(M_1 - M_2))}.$$

Let $g_*(s) = \inf\{\lambda : d_g(\lambda) \geq s\}$. From the previous inequality we obtain

$$g_*(s) \geq \left(\frac{sr^d d}{(4et)^d a} \right)^{c(M_1 - M_2)/d}.$$

In particular,

$$\begin{aligned} \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} g_t d\mathcal{H}^d &= \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^{\mathcal{H}^d(K_{x,t})} g_*(s) ds \geq \\ \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^{\mathcal{H}^d(K_{x,t})} \left(\frac{sr^d d}{(4et)^d a} \right)^{c(M_1 - M_2)/d} ds &\geq \frac{1}{1 + c(M_1 - M_2)/d} \left(\frac{r^d db}{(4e)^d a} \right)^{c(M_1 - M_2)/d}. \end{aligned}$$

Here we used inequality $\mathcal{H}^d(x, t) \geq bt^d$. Thus we obtain

$$\sup_{K_{x,t}} e^f \leq (1 + c(M_1 - M_2)/d) \left(\frac{(4e)^d a}{r^d db} \right)^{c(M_1 - M_2)/d} \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^f d\mathcal{H}^d \quad (3.8)$$

which implies the required local reverse Hölder inequality. In the general case, we cover again K by a finite number of open disks $\mathbb{D}(x_i, R)$, $i = 1, \dots, N$ such that f is defined in the union of these disks, the set $\cup_{i=1}^N \mathbb{D}(x_i, R/2)$ also covers K and any disk of radius $\leq R/4$ centered at a point of K belongs to one of $\mathbb{D}(x_i, R/2)$. Then the reverse Hölder inequality of the form (3.8) holds for any $K_{x,t} = \mathbb{D}(x, t) \cap K$, $x \in K$, $t \leq R/4$. Assume now that $t > R/4$ and set

$$m := \inf_{x \in K, t > R/4} \left\{ \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^f d\mathcal{H}^d \right\}.$$

Then $m > 0$. Indeed, let $x_i, t_i > R/4$, be a sequence for which the expression on the right above converges to m . Without loss of generality we may assume also that x_i tends to $x \in K$ and t_i tends to $t \geq R/4$. Then there is i_0 such that for any $i \geq i_0$, the ball K_{x_i, t_i} contains $K_{x, R/8}$. Note that $\sup_{K_{x, R/8}} e^f > 0$ because $K_{x, R/8}$ is not a polar set. Then inequality (3.8) applied to $K_{x, R/8}$ and the d -regularity of K show that

$$m \geq \frac{C}{\mathcal{H}^d(K_{x, R/8})} \int_{K_{x, R/8}} e^f d\mathcal{H}^d > 0$$

for a constant $C := C(K)$. Finally, since $\sup_{K_{x,t}} e^f \leq M := \sup_K e^f < \infty$, inequality (3.8) for $t > R/4$ is valid with the constant M/m .

The proof of the theorem is complete. \square

Corollary 3.7 *Assume that a subharmonic function f defined on \mathbb{C} satisfies*

$$f(z) \leq c' + \log(1 + |z|) \quad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in \mathcal{A}(d, a, b)$. Then for $e^f|_S$ the reverse Hölder inequality (3.8) holds with the constant $\frac{c_1}{d} \left(\frac{a}{db}\right)^{c_2/d}$, where c_1, c_2 are absolute positive constants.

Proof. The proof follows directly from the Bernstein-Walsh inequality (3.7) and Theorem 1.4. \square

4. Multidimensional Case.

In this part we prove the results of section 1.2. Let k be the number of zeros of e^{2u} in $B_{1/2}$ counting with their multiplicities (see definition in section 1.2). Here $u = \frac{1}{2} \log(|f_1|^2 + \dots + |f_n|^2)$ satisfies inequalities (1.5). Below we estimate k by M, n, r only.

Theorem 4.1 *Given $H > 0, d > 0$ there exists a system of Euclidean balls such that*

$$\sum r_j^d \leq \frac{(2H)^d}{d}$$

where r_j are radii of these balls, and

$$u(z) \geq -M + k \log \frac{H}{e}$$

everywhere outside these balls in $B_{1/2}$.

Proof. Let ξ_1, \dots, ξ_k be zeros of e^u in $B_{1/2}$. We begin with the following

Lemma 4.2

$$-M + \sum_{i=1}^k \log |z - \xi_i| \leq u(z) \quad (z \in B_{1/2}).$$

Proof. Without loss of generality we may assume that each of zeros of the system $F = 0$ is of multiplicity 1. In fact, according to our assumptions image $F(B_{1/2}) \subset \mathbb{C}^n$ is of complex dimension n . In particular, by Sard's theorem we can approximate F by maps $F_c = F - c$ where c is a regular value of F close to $0 \in \mathbb{C}^n$ and $F_c^{-1}(0)$ is a family of zeros of multiplicity 1. Then we prove the lemma for $\log |F_c|$ and going to the limit as $c \rightarrow 0$ obtain the required statement. Further, observe that u satisfies the complex Monge-Ampere equation everywhere in $B_{1/2} \setminus \{\xi_1, \dots, \xi_k\}$. In fact, $u = \frac{1}{2} F^* U$, where $U = \log(\sum_{i=1}^n |z_i|^2)$ satisfies the Monge-Ampere equation in $\mathbb{C}^n \setminus \{0\}$. Since F is holomorphic, u satisfies the required equation on $B_{1/2} \setminus F^{-1}(0)$. We recall the following result from [BT]:

Assume that u_1, u_2 are continuous plurisubharmonic functions in a bounded domain D with a compact boundary K . Assume also that $u_1 \geq u_2$ on K and u_1 satisfies the complex Monge-Ampere equation in an open neighbourhood of D . Then $u_1 \geq u_2$ everywhere on D .

Let $g_n = -M + (1 + 1/n) \sum_{i=1}^k \log |z - \xi_i|$. Since by the assumption ξ_i is a simple zero of F , for any i there is a ball $B_{r_{n,i}}$ of small radius $r_{n,i}$ centered at ξ_i such that $g_n \leq u$ on its boundary. Without loss of generality we may assume that these balls are pairwise disjoint and $r_{n,i} \rightarrow 0$ as $n \rightarrow \infty$. Moreover, by definition $g_n \leq u$ on $S_{1/2}$. Then according to the above maximal principle, $g_n \leq u$ in $B_{1/2} \setminus (\cup_i B_{r_{n,i}})$. It remains to take the limit as $n \rightarrow \infty$ to obtain by continuity $g \leq u$ in $B_{1/2}$ where $g = -M + \sum_{i=1}^k \log |z - \xi_i|$.

The lemma is proved. \square

We now apply Theorem 2.3 to the function g with $X = \mathbb{C}^n$, $d(x, y) = |x - y|$ and $\mu = \sum_{i=1}^k \delta_{\xi_i}$. Then we obtain

Given $H > 0$, $d > 0$, there exists a system of Euclidean balls such that

$$\sum r_j^d \leq \frac{(2H)^d}{d}$$

where r_j are radii of these balls, and

$$g(z) \geq -M + k \log \frac{H}{e}$$

everywhere outside these balls. Taking into account that $u \geq g$ in $B_{1/2}$ we obtain the required statement.

The proof of Theorem 4.1 is complete. \square

Remark 4.3 In the inequality of Theorem 4.1 we can take any $p \geq k$ instead of k . We obtain this replacing inequality of Lemma 4.2 by

$$-M + \frac{p}{k} \sum_{i=1}^k \log |z - \xi_i| \leq u(z) \quad (z \in B_{1/2})$$

and then repeating the arguments of the proof of Theorem 4.1 applied to $\frac{k}{p}u$.

We now estimate the number of zeros k .

Lemma 4.4 *Under assumptions of Theorem 4.1*

$$k \leq c(r, n)e^{(2n-1)M}. \quad (4.1)$$

Proof. Let $h = \log(|z_1|^2 + \dots + |z_n|^2)$. Consider the differential form $\omega = C(n)(\bar{\partial}h) \wedge (\bar{\partial}\partial h)^{n-1}$. Then we have $d\omega = 0$ on $\mathbb{C}^n \setminus \{0\}$ and for some $C(n) \in \mathbb{C}$ the Bochner-Martinelli formula is valid

$$\phi(0) = \int_{\partial D} \phi(\xi) \omega.$$

Here D is a domain containing 0 with a smooth boundary ∂D and ϕ is holomorphic in an open neighbourhood of \bar{D} . Consider now the form $F^*\omega$ in B_r . Since F is a holomorphic map, $F^*\omega = C(n)(\bar{\partial}F^*h) \wedge (\bar{\partial}\partial F^*h)^{n-1}$ and $d(F^*\omega) = 0$ on $B_{1/2} \setminus F^{-1}(0)$. In particular, by Stocks' theorem $\int_{S_{1/2}} F^*\omega = \sum_{i=1}^k \int_{S_i} F^*\omega$, where S_i is a sphere of a small radius centered at ξ_i and $S_{1/2} = \partial B_{1/2}$. Assume without loss of generality that 0 is a regular value of F , i.e., there are small neighbourhoods of ξ_1, \dots, ξ_k such that F maps them biholomorphically to a ball centered at 0. Assume also that these neighbourhoods contain S_1, \dots, S_k . Doing in each of these neighbourhoods a holomorphic change of variables and then applying the Bochner-Martinelli formula we obtain $\int_{S_{1/2}} F^*\omega = k$. Note that

$$F^*\omega = C'(n) \frac{F^*\sigma}{|F|^{2n}},$$

where $\sigma = \sum_{i=1}^n (-1)^{i-1} \bar{z}_i d\bar{z}_1 \wedge \dots \wedge \bar{z}_n \wedge dz$ and $dz = dz_1 \wedge \dots \wedge dz_n$ (see [GH]). From here, Cauchy's inequalities for the first derivatives of a holomorphic function and the estimate $|F| \geq e^{-M}$ on $S_{1/2}$ we finally get

$$k \leq c(r, n)e^{(2n-1)M}. \quad \square$$

Remark 4.5 Unlike the one-dimensional case, the global Cartan's estimate of Theorem 4.1 do not imply similar local estimates in each ball inside of $B_{1/2}$ (consider, e.g., function $\log(|z_1|^2 + |z_2|^4)$). However, under assumptions of Theorem 1.6 the multidimensional case is similar to one-dimensional.

Proof of Theorem 1.6. Assume that all zeros of $|F|^2$ are elliptic.

Proposition 4.6 *There is a constant $c = c(r, F) > 0$ such that for any ball $B(x, t)$ satisfying $B(x, t) \subset B(x, 4r^2t) \subset B_{1/2}$ and any $H > 0, d > 0$ there is a system of balls such that*

$$\sum r_j^d \leq \frac{(8rtH)^d}{d},$$

where r_j are radii of these balls, and

$$\log |F| \geq \sup_{B(x, t)} \log |F| + k \log \frac{cH}{e}$$

outside these balls in $B(x, t)$.

Proof.

Set

$$f_{x,t}(z) = \log |F(z)| - \sup_{B(x,4r^2t)} \log |F|$$

and

$$H_{x,t} = \sup_{t \leq p \leq 2rt} \inf_{z \in S(x,p)} f_{x,t}(z),$$

where $S(x,p) := \{z \in \mathbb{C}^n : |z - x| = p\}$. Let $K := \{(x,t)\}$ be the set of centers and radii of balls satisfying conditions of Proposition 4.6.

Lemma 4.7 $C := \sup_{(x,t) \in K} H_{x,t} > -\infty$.

Proof. Assume that $\{(x_n, t_n)\}_{n \geq 1} \subset K$ is a sequence satisfying

$$\lim_{n \rightarrow \infty} H_{x_n, t_n} = C.$$

Without loss of generality we may assume that $B(x_n, t_n) \rightarrow B(x^*, t^*)$ in the Hausdorff metric. Further, consider the following cases.

(1) $t^* > 0$. Then $C = H_{x^*, t^*}$ by continuity. Clearly, $C > -\infty$ because F has only finite number of zeros in $B(x^*, 2rt^*)$.

(2) $t^* = 0$. If x^* is not a zero of F then $H_{x^*, t^*} = 0$ by continuity. Assume now that x^* is a zero of F . Then by ellipticity of x^* the equality

$$\log |F(x^* + t\omega)| = s \log t + \log f(\omega) + o(t) \quad (0 \leq t \leq t_0, \omega \in S^{2N-1}, s \leq k)$$

holds for a sufficiently small t_0 . Here $0 < \inf_{S^{2N-1}} f \leq \sup_{S^{2N-1}} f < \infty$. From this representation it follows that it suffices to check the lemma in this case for $|F| = t$, $t \leq t_0$, and $x^* = 0$. Then a straightforward computation gets

$$\sup_{B(x_n, 2rt_n)} \log t = \log(|x_n| + 4r^2 t_n)$$

and so

$$H_{x_n, t_n} = \begin{cases} \log \frac{|x_n| - t_n}{|x_n| + 4r^2 t_n} & (0 \notin B(x_n, \frac{(2r+1)t_n}{2})); \\ \log \frac{2rt_n - |x_n|}{|x_n| + 4r^2 t_n} & (0 \in B(x_n, \frac{(2r+1)t_n}{2})). \end{cases}$$

In the first case H_{x_n, t_n} is a monotonically increasing in x_n function. Thus

$$H_{x_n, t_n} \geq H_{(2r+1)t_n/2, t_n} = \log \frac{2r-1}{8r^2 + 2r + 1} > -\infty$$

because $r > 1/2$. In the second case H_{x_n, t_n} is a monotonically decreasing in x_n function. This implies

$$H_{x_n, t_n} \geq H_{(2r+1)t_n/2, t_n} = \log \frac{2r-1}{8r^2 + 2r + 1} > -\infty.$$

The proof of the lemma is complete. \square

We proceed with the proof of Proposition 4.6. According to Lemma 4.7 there is a sphere $S(x, p)$, $t \leq p \leq 2rt$, such that $\inf_{S(x, p)} f_{x, t} \geq C > -\infty$. In addition, by conditions of the theorem $\sup_{B(x, 2rp)} f_{x, t} \leq 0$. We set $F'(z) = f_{x, t}(2zp)$, $|z| \leq r$. Then F' satisfies inequalities (1.5). Applying Theorem 4.1 to F' and going back to the ball $B(x, t) \subset B(x, p)$ we obtain

Given $H > 0, d > 0$, there exists a system of Euclidean balls such that

$$\sum r_j^d \leq \frac{(8rtH)^d}{d}$$

where r_j are radii of these balls, and

$$\log |F| \geq \sup_{B(x, 4r^2t)} \log |F| + C + k \log \frac{H}{e} \geq \sup_{B(x, t)} \log |F| + k \log \frac{e^C H}{e}$$

outside these balls in $B(x, t)$. We used here that $C \leq 0$.

The proposition is proved. \square

Proofs of Theorems 1.6 and 1.7. Proofs of these results repeat word-for-word proofs of Theorems 1.1, 1.3 and 1.4 and might be left to the reader. \square

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